Suggested Solutions of HW5

Ex 16.4

39. (a) For $\nabla f = \left(\frac{2x}{x^2+y^2}\right)i + \left(\frac{2y}{x^2+y^2}\right)j$, $M = \frac{2x}{x^2+y^2}$, $N = \frac{2y}{x^2+y^2}$

Because they are discontinuous at $(0,0)$, we can not use Green's Thm.

Then let $x = a\cos t$, $y = a\sin t$, so $\int_C \nabla f \cdot n\,ds = \int_C M\,dy - N\,dx$

$= \int_0^{2\pi} \left(\frac{2}{a}\cos t\right)(a\cos t) - \left(\frac{2}{a}\sin t\right)(-a\sin t)\,dt = 4\pi$ for any circle C centered at $(0,0)$ traversed counterclockwise.

And $\int_C \nabla f \cdot n\,ds = -4\pi$ if C is traversed clockwise.

(b) If $(0,0) \notin K$, by Green's Thm: $\int_C \nabla f \cdot n\,ds = \int_C M\,dy - N\,dx$

$= \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)dx\,dy = \iint_R \left(\frac{2(y^2-x^2)}{(x^2+y^2)^2} + \frac{2(x^2-y^2)}{(x^2+y^2)^2}\right)dx\,dy = 0.$

If $(0,0) \in K$, let C be a small circle centered at $(0,0)$ and $C \subset K$. Thus by Green's Thm $\int_K M\,dy - N\,dx + \int_C M\,dy - N\,dx$

$= \iint_{K \setminus C} \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)dx\,dy = 0$, where K is counterclockwise and C is clockwise

Hence by (a) $\int_K \nabla f \cdot n\,ds = 4\pi$. #

Practice Ex:

37. (a) Now $r = (e^t\cos t)i + (e^t\sin t)j$, then $x = e^t\cos t$, $y = e^t\sin t$ from $(1,0)$ to $(e^{2\pi},0)$

So $t \in [0, 2\pi]$. By calculation, $F \cdot \frac{dr}{dt} = e^{-t}$

Thus Work $= \int_0^{2\pi} e^{-t}\,dt = 1 - e^{-2\pi}$

(b) For $F = \frac{xi+yj}{(x^2+y^2)^{\frac{3}{2}}}$, so $f(x,y,z) = -(x^2+y^2)^{-\frac{1}{2}} + g(y,z)$.

And $\frac{\partial f}{\partial y} = \frac{y}{(x^2+y^2)^{\frac{3}{2}}} + \frac{\partial g}{\partial y} = \frac{y}{(x^2+y^2)^{\frac{3}{2}}}$, so $\frac{\partial g}{\partial y} = $ Constant.

Then $f = -(x^2+y^2)^{-\frac{1}{2}}$ is a potential for F.

Thus $\int_C F \cdot dr = f(e^{2\pi},0) - f(1,0) = 1 - e^{-2\pi}$.

50. For $M = y - 6x^2$, $N = x + y^2$. So Flux $= \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)dx\,dy = \iint_R (-12x+2y)\,dx\,dy$

$= \int_0^1 \int_y^1 (-12x+2y)\,dx\,dy = -\frac{11}{3}$. And Circulation $= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dx\,dy = 0.$

Additional Ex.:

2. $dx = (-2\sin t - 2\sin 2t)dt$ and $dy = (2\cos t - 2\cos 2t)dt$.

Then Area $= \frac{1}{2}\oint_C x\,dy - y\,dx = \frac{1}{2}\int_0^{2\pi} [2 - 2(\cos t\cos 2t - \sin t\sin 2t)]dt$

$= \frac{1}{2}\int_0^{2\pi}(2 - 2\cos 3t)dt = 2\pi$.

4. $dx = (-2a\sin t - 2a\cos 2t)\,dt$ and $dy = b\cos t\,dt$.

Then Area $= \frac{1}{2}\oint_C x\,dy - y\,dx = \frac{1}{2}\int_0^{2\pi} [(2ab\cos^2 t - ab\cos t\sin 2t) - (-2ab\sin^2 t - 2ab\sin t\cos 2t)]dt$

$= \frac{1}{2}\int_0^{2\pi}(2ab + 2ab\cos^2 t\sin t - 2ab\sin t)dt = 2\pi ab$.

Ex. 15.5

22. (a) $\int_0^1\int_0^1\int_{-1}^{-\sqrt{z}} dy\,dz\,dx$ (b) $\int_0^1\int_0^1\int_{-1}^{-\sqrt{z}} dy\,dx\,dz$ (c) $\int_0^1\int_{-1}^{-\sqrt{z}}\int_0^1 dx\,dy\,dz$

(d) $\int_{-1}^0\int_0^{y^2}\int_0^1 dx\,dz\,dy$ (e) $\int_{-1}^0\int_0^1\int_0^{y^2} dz\,dx\,dy$

24. $V = \int_0^1\int_0^{1-x}\int_0^{2-2x} dy\,dz\,dx = \int_0^1\int_0^{1-x}(2-2x)\,dz\,dx = \int_0^1 1-x^2\,dx = \frac{2}{3}$.

26. $V = 2\int_0^1\int_{-\sqrt{1-x^2}}^0\int_0^{-y} dz\,dy\,dx = -2\int_0^1\int_{-\sqrt{1-x^2}}^0 y\,dy\,dx = \frac{2}{3}$

28. $V = \int_0^1\int_0^{1-x}\int_0^{\cos\frac{\pi x}{2}} dz\,dy\,dx = \int_0^1\int_0^{1-x}\cos\frac{\pi x}{2}\,dy\,dx = \int_0^1\cos\frac{\pi x}{2}(1-x)\,dx$

$= \int_0^1\cos\frac{\pi x}{2}\,dx - \int_0^1 x\cos\frac{\pi x}{2}\,dx = \frac{2}{\pi} - \frac{4}{\pi^2}\int_0^{\frac{\pi}{2}} u\cos u\,du = \frac{2}{\pi} - \frac{4}{\pi^2}[\cos u + u\sin u]_0^{\frac{\pi}{2}}$

$= \frac{4}{\pi^2}$

36. $V = 2\int_0^1\int_0^{1-y^2}\int_0^{x^2+y^2} dz\,dx\,dy = 2\int_0^1\int_0^{1-y^2}(x^2+y^2)\,dx\,dy = 2\int_0^1(1-y^2)[\frac{1}{3}(1-y^2)^2 + y^2]dy$

$= \frac{2}{3}\int_0^1(1-y^6)\,dy = \frac{4}{7}$.